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Picture fuzzy aggregation approach with application to third-party logistic provider selection process

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ABSTRACT

Picture fuzzy sets (PFSs) are frequently composed of positive, neutral, and negative memberships and have the benefit of precisely capturing the preferences of decision makers (DMs). This paper introduces innovative picture fuzzy aggregation operators (AOs) based on fundamental operations, which have a number of advantages when dealing with real-world scenarios. Unpredictability and fuzziness coexist in decision-making analysis due to the complexity of the decision-making environment. Picture fuzzy numbers (PFNs) outperform intuitionistic fuzzy numbers (IFNs) when dealing with unclear input. This study proposes two AOs: picture fuzzy hybrid weighted arithmetic geometric aggregation (PFHWAGA) operator and picture fuzzy hybrid ordered weighted arithmetic geometric aggregation (PFHOWAGA) operator. The proposed operators outperform the current PFN-defined operators. The proposed operator is utilised in the multi-criteria decision-making (MCDM) process to third-party logistic provider selection.

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1 Introduction

MCDM is a frequently used cognitive technique whose major objective is to select the optimal choice from a finite set of alternatives with the assistance of the decision maker's (DM's) expert judgement. To solve these issues, Zadeh (1965) developed the field of fuzzy set (FS) theory, which uses a mathematical paradigm to explain ambiguity. Atanassov (1986) widened the scope of this fiction by introducing the concept of intuitionistic fuzzy set (IFS) theory. It is critical to recognize that FS may be described in terms of membership characteristics, whereas IFS can be expressed in terms of membership and non-membership characteristics. While IFS has been widely utilised to address significant challenges, there are certain cases that IFSs cannot handle. Assume that the human view on voting includes additional responses such as yes, abstain, no, and refuse, which conventional FS and IFS cannot adequately reflect. Cuong (2013a,b) pioneered the notion of picture fuzzy set (PFS) to solve these issues. Each element in PFS has a positive membership grade (PMG), a neutral membership grade (NuMG), and a negative membership grade (NgMG), all of which have values between $[0, 1]$.

Data analysis is essential for decision-making in the fields of organisational, sociological, therapeutic, scientific, cognitive, and machine intelligence, among other fields of study. Understanding of the alternative has traditionally been referred to as a crisp number or a linguist number, respectively. Unfortunately, because of the volatility of the data, it cannot simply be gathered and analysed. However, in practise, AOs are critical in the context of MCDM issues since their primary purpose is to agglomerate a large number of inputs into a single value. The famous Maclaurin symmetric mean (MSM) AOs linked to IFSs were introduced by Liu and Qin (2017). Gul (20221) pioneered the concept of

Fermatean fuzzy SAW, VIKOR and ARAS, which he applied to the COVID-19 testing laboratory prediction phase. MCDM technique based on fuzzy rough sets was introduced by Ye et al. (2021); Mu et al. (2021).

Garg (2017) proposed many averaging and ordered averaging AOs for PFNs. Tian et al. (2019) defined some picture fuzzy power Choquet ordered geometric AOs & picture fuzzy power shapley Choquet ordered geometric AOs with shapley fuzzy measures based MCDM. Wang et al.(2020) proposed hotel building energy efficiency retrofit project selection under PFSs. Wang et al. (2019) introduced Muirhead mean AOs for PFNs. initiated the concept of Hamacher AOs and Jana *et al.* Jana et al. (2019) proposed Dombi AOs for PFNs. Farid and Riaz (2021) devel-oped several new Einstein interactive geometric AOs for q-rung orthopair fuzzy numbers. Riaz and Farid (2022a) proposed some proportional distribution based spherical AOs. Farid et al. (2022) introduced some AOs for the thermal power equipment supplier selection. Ali et al. (2021) and Ashraf et al. (2022) proposed some AOs for interval-valued picture fuzzy set. Kazemitash et al. (2021) and Bozanic et al. (2021) gave some ideas related to some different extensions of fuzzy set. Riaz and Farid (2022b) developed some fairly AOs for PFSs. Riaz et al. (2022) proposed some Frank AOs for interval-valued linear Diophantine fuzzy set. For other terminologies not discussed in the paper, the readers are referred to Mukhametzyanov (2021); Sahu et al. (2021); Karamasa et al. (2021); Riaz et al. (2020).

The remainder of this article will be organised in the following manner. Section 2 discusses several important PFS concepts. Section 3 considered a number of hybrid AOs for PFSs. Section 4 describes a technique for solving MCDM issues with new AOs. Section 5 is a call for details on proposed AOs. Section 6 concludes with some final remarks and future recommendations.

2 Preliminaries

In this part, we will go through the fundamentals of FSs, IFs, and PFS-sets.

Definition 2.1. Zadeh (1965) Let $\mathfrak{X}^{\Theta}$ be the reference set. A FS $\mathfrak{H}$ is

$$\mathfrak{H} = \{< \wp^{\Lambda}, \mu^{\gimel}{}_{\mathfrak{H}}(\wp^{\Lambda}) >: \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$$

where $\mu^{\gimel}{}_{\mathfrak{H}} : \mathfrak{X}^{\Theta} \to [0, 1]$ is the MSD of $\mathfrak{H}$ which assigns a single real value to each alternative in the unit closed interval $[0, 1]$.

Definition 2.2. Atanassov (1986) An IFS is

$$T = \{< \wp^{\Lambda}, \mu^{\gimel}{}_{T}(\wp^{\Lambda}), \nu^{\gimel}{}_{T}(\wp^{\Lambda}) >: \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$$

which is represented by MSD $\mu^{\gimel}{}_{T}(\wp^{\Lambda}) : \mathfrak{X}^{\Theta} \to [0, 1]$ and N-MSD $\nu^{\gimel}{}_{T}(\wp^{\Lambda}) : \mathfrak{X}^{\Theta} \to [0, 1]$ with the constraint $0 \leq \mu^{\gimel}{}_{T}(\wp^{\Lambda}) + \nu^{\gimel}{}_{T}(\wp^{\Lambda}) \leq 1, \forall \wp^{\Lambda} \in \mathfrak{X}^{\Theta}$.

Definition 2.3. Cuong (2013a,b) A PFS in a universe $\mathfrak{X}^{\Theta}$ is

$$\vartheta^{\aleph} = \{< \wp^{\Lambda}, \mu^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}), \tau^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}) >: \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$$

where $\mu^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}) : \mathfrak{X}^{\Theta} \to [0, 1]$ shows the PMSD, $\nu^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}) : \mathfrak{X}^{\Theta} \to [0, 1]$ shows the NuMSD, $\tau^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}) : \mathfrak{X}^{\Theta} \to [0, 1]$ shows the NgMSD of the element $\wp^{\Lambda} \in \mathfrak{X}^{\Theta}$ to the set $\vartheta^{\aleph}$, respectively, with the condition that $0 \leq \mu^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}) + \nu^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}) + \tau^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}) \leq 1$.
A basic element of the form $\langle \mu^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}), \tau^{\gimel}{}_{\vartheta^{\aleph}}(\wp^{\Lambda}) \rangle$ in a PFS $\vartheta^{\aleph}$ is called picture fuzzy number (PFN). It is denoted by $\vartheta^{\aleph} = \langle \mu^{\gimel}{}_{\vartheta^{\aleph}}, \nu^{\gimel}{}_{\vartheta^{\aleph}}, \tau^{\gimel}{}_{\vartheta^{\aleph}} \rangle$.

2.1 Operational laws for PFSs (See Cuong (2013a,2013b), Wei (2017,2018), Garg (2017)).

Definition 2.4. Cuong (2013a,b) Let $\vartheta^{\aleph}{}_1 = \langle \mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}) \rangle$ and $\vartheta^{\aleph}{}_2 = \langle \mu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}), \tau^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}) \rangle$ be PFSs on a $\mathfrak{X}^{\Theta}$. Then
(1) $\vartheta^{\bar{\aleph}}{}_1 = \langle \tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}) \rangle$.
(2) $\vartheta^{\aleph}{}_1 \subseteq \vartheta^{\aleph}{}_2$ iff $\mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}) \leqslant \mu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}) \leqslant \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})$ and $\tau^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}) \leqslant \tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})$.
(3)$\vartheta^{\aleph}{}_1 = \vartheta^{\aleph}{}_2$ iff $\mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}) = \mu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}) = \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})$ and $\tau^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}) = \tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})$.
(4) $\vartheta^{\aleph}{}_1 \cup \vartheta^{\aleph}{}_2 = \{\langle \wp^{\Lambda}, max\{\mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \mu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})\}, min\{\nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})\}\rangle, min\{\tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \tau^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})\}\rangle \mid \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$.
(5) $\vartheta^{\aleph}{}_1 \cap \vartheta^{\aleph}{}_2 = \{\langle \wp^{\Lambda}, min\{\mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \mu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})\}, max\{\nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})\}\rangle, max\{\tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}), \tau^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})\}\rangle \mid \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$.
(6) $\vartheta^{\aleph}{}_1 + \vartheta^{\aleph}{}_2 = \{\langle \wp^{\Lambda}, (\mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}) + \mu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}) - \mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})\mu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})\nu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}), \tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})\tau^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})\rangle \mid \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$.
(7) $\vartheta^{\aleph}{}_1 . \vartheta^{\aleph}{}_2 = \{\langle \wp^{\Lambda}, (\mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})\mu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}) + \nu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}) - \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})\nu^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}),$
$\tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}) + \tau^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda}) - \tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})\tau^{\gimel}{}_{\vartheta^{\aleph}{}_2}(\wp^{\Lambda})\rangle \mid \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$.
(8) $\beth\vartheta^{\aleph}{}_1 = \{\langle \wp^{\Lambda}, (1 - (1 - \mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}))^{\beth})), \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})^{\beth}, \tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})^{\beth} \mid \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\rangle\}$.
(9) $\vartheta^{\aleph}{}_1^{\beth} = \{\langle \wp^{\Lambda}, \mu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda})^{\beth}, (1 - (1 - \nu^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}))^{\beth})), (1 - (1 - \tau^{\gimel}{}_{\vartheta^{\aleph}{}_1}(\wp^{\Lambda}))^{\beth}))\rangle \mid \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$.

Theorem 2.5. Let $P^{\gimel}$, $B^{\gimel}$ and $C^{\gimel}$ be any PFSs over the reference set $\mathfrak{X}^{\Theta}$. Let $\widetilde{U}$ be absolute PFS and $\widetilde{\varnothing}$ be the null PFS. Then

(i) $P^{\gimel} \cup P^{\gimel} = P^{\gimel}$.
(ii) $P^{\gimel} \cap P^{\gimel} = P^{\gimel}$.
(iii) $(P^{\gimel} \cup B^{\gimel}) \cup C^{\gimel} = P^{\gimel} \cup (B^{\gimel} \cup C^{\gimel})$.
(iv) $(P^{\gimel} \cap B^{\gimel}) \cap C^{\gimel} = P^{\gimel} \cap (B^{\gimel} \cap C^{\gimel})$.
(v) $P^{\gimel} \cup (B^{\gimel} \cap C^{\gimel}) = (P^{\gimel} \cup B^{\gimel}) \cap (P^{\gimel} \cup C^{\gimel})$.
(vi) $P^{\gimel} \cap (B^{\gimel} \cup C^{\gimel}) = (P^{\gimel} \cap B^{\gimel}) \cup (P^{\gimel} \cap C^{\gimel})$.
(vii) $P^{\gimel} \cup \widetilde{\varnothing} = P^{\gimel}$ and $P^{\gimel} \cap \widetilde{\varnothing} = \widetilde{\varnothing}$
(viii) $P^{\gimel} \cup \widetilde{U} = \widetilde{U}$ and $P^{\gimel} \cap \widetilde{U} = P^{\gimel}$
(ix) $(P^{\gimel c})^{c} = P^{\gimel}$
(x) $\widetilde{U}^{c} = \widetilde{\varnothing}$ and $\widetilde{\varnothing}^{c} = \widetilde{U}$

Theorem 2.6. *Let $P^{\gimel}$ and $B^{\gimel}$ be two PFSs over the reference set $\mathfrak{X}^{\Theta}$. Then*

(a) $(P^{\gimel} \cup B^{\gimel})^{c} = P^{\gimel c} \cap B^{\gimel c}$, *and*
(b) $(P^{\gimel} \cap B^{\gimel})^{c} = P^{\gimel c} \cup B^{\gimel c}$.

2.2 Operational laws for PFNs (See Cuong (2013a,2013b), Wei (2017,2018), Garg (2017)).

Definition 2.7. Cuong (2013a) Suppose $\Upsilon^{\varsigma}{}_1 = \langle \mu^{\gimel}{}_1, \nu^{\gimel}{}_1, \tau^{\gimel}{}_1 \rangle$ and $\Upsilon^{\varsigma}{}_2 = \langle \mu^{\gimel}{}_2, \nu^{\gimel}{}_2, \tau^{\gimel}{}_2 \rangle$ are the two PFNs. Then
(1) $\bar{\Upsilon^{\varsigma}}_1 = \langle \tau^{\gimel}{}_1, \nu^{\gimel}{}_1, \mu^{\gimel}{}_1 \rangle$
(2) $\Upsilon^{\varsigma}{}_1 \vee \Upsilon^{\varsigma}{}_2 = \langle max\{\mu^{\gimel}{}_1, \mu^{\gimel}{}_1\}, min\{\nu^{\gimel}{}_2, \nu^{\gimel}{}_2\}, min\{\tau^{\gimel}{}_2, \tau^{\gimel}{}_2\} \rangle$
(3) $\Upsilon^{\varsigma}{}_1 \wedge \Upsilon^{\varsigma}{}_2 = \langle min\{\mu^{\gimel}{}_1, \mu^{\gimel}{}_1\}, max\{\nu^{\gimel}{}_2, \nu^{\gimel}{}_2\}, max\{\tau^{\gimel}{}_2, \tau^{\gimel}{}_2\} \rangle$
(4) $\Upsilon^{\varsigma}{}_1 \oplus \Upsilon^{\varsigma}{}_2 = \langle \mu^{\gimel}{}_1 + \mu^{\gimel}{}_2 - \mu^{\gimel}{}_1 \mu^{\gimel}{}_2, \nu^{\gimel}{}_1 \nu^{\gimel}{}_2, \tau^{\gimel}{}_1 \tau^{\gimel}{}_2 \rangle$
(5) $\Upsilon^{\varsigma}{}_1 \otimes \Upsilon^{\varsigma}{}_2 = \langle \mu^{\gimel}{}_1 \mu^{\gimel}{}_2, \nu^{\gimel}{}_1 + \nu^{\gimel}{}_2 - \nu^{\gimel}{}_1 \nu^{\gimel}{}_2, \tau^{\gimel}{}_1 + \tau^{\gimel}{}_2 - \tau^{\gimel}{}_1 \tau^{\gimel}{}_2 \rangle$
(6) $\beth \Upsilon^{\varsigma}{}_1 = \langle (1 - (1 - \mu^{\gimel}{}_1)^{\beth}), \nu^{\gimel \beth}_1, \tau^{\gimel \beth}_1 \rangle$
(7) $\Upsilon^{\varsigma \Upsilon^{\varsigma}{}_1}_1 = \langle \mu^{\gimel \beth}_1, (1 - (1 - \nu^{\gimel}{}_1)^{\beth}), (1 - (1 - \tau^{\gimel}{}_1)^{\beth}) \rangle$

Theorem 2.8. Cuong (2013b) Suppose $\Upsilon^{\varsigma}{}_1 = \langle \mu^{\gimel}{}_1, \nu^{\gimel}{}_1, \tau^{\gimel}{}_1 \rangle$ and $\Upsilon^{\varsigma}{}_2 = \langle \mu^{\gimel}{}_2, \nu^{\gimel}{}_2, \tau^{\gimel}{}_2 \rangle$ are the two PFNs on a $\mathfrak{X}^{\Theta}$, and $n_1, n_2 > 0$, then
(1) $\Upsilon^{\varsigma}{}_1 \oplus \Upsilon^{\varsigma}{}_2 = \Upsilon^{\varsigma}{}_2 \oplus \Upsilon^{\varsigma}{}_1$

(2) $\Upsilon^{\varsigma}{}_1 \otimes \Upsilon^{\varsigma}{}_2 = \Upsilon^{\varsigma}{}_2 \otimes \Upsilon^{\varsigma}{}_1$

(3) $n(\Upsilon^{\varsigma}{}_1 \oplus \Upsilon^{\varsigma}{}_2) = n\Upsilon^{\varsigma}{}_1 \oplus n\Upsilon^{\varsigma}{}_2$

(4) $n_1 \Upsilon^{\varsigma}{}_1 \oplus n_2 \Upsilon^{\varsigma}{}_2 = (n_1 + n_2)\Upsilon^{\varsigma}{}_1$

(5) $\Upsilon^{\varsigma n_1}_1 \otimes \Upsilon^{\varsigma n_2}_1 = \Upsilon^{\varsigma n_1 + n_2}_1$

(6) $\Upsilon^{\varsigma n}_1 \otimes \Upsilon^{\varsigma n}_2 = (\Upsilon^{\varsigma}{}_1 \otimes \Upsilon^{\varsigma}{}_2)^n$

Definition 2.9. Garg (2017) Let $\vartheta^{\aleph} = \langle \mu^{\gimel}, \nu^{\gimel}, \tau^{\gimel} \rangle$ be the PFN, then a score function (SF) $\tilde{\daleth}$ of $\vartheta^{\aleph}$ is given as

$$\tilde{\daleth}(\vartheta^{\aleph}) = \mu^{\gimel} - \nu^{\gimel} - \tau^{\gimel}$$

$\tilde{\daleth}(\vartheta^{\aleph}) \in [-1, 1]$. If the SF is high, the PFN is also significant. Unfortunately, in the many situations of PFN, the SF is ineffective. We employ another function called the accuracy function (AF) to tackle this difficulty.

Definition 2.10. Wei (2017) & Garg (2017) Let $\vartheta^{\aleph} = \langle \mu^{\gimel}, \nu^{\gimel}, \tau^{\gimel} \rangle$ be the PFN. Then the AF $\amalg$ of $\vartheta^{\aleph}$ is defined as

$$\amalg(\vartheta^{\aleph}) = \mu^{\gimel} + \nu^{\gimel} + \tau^{\gimel}$$

where $\amalg(\vartheta^{\aleph}) \in [0, 1]$. If the AF is high, the PFN is also significant.

Definition 2.11. Let $s = \langle \mu^{\gimel}{}_s, \nu^{\gimel}{}_s, \tau^{\gimel}{}_s \rangle$ and $t = \langle \mu^{\gimel}{}_t, \nu^{\gimel}{}_t, \tau^{\gimel}{}_t \rangle$ be any two PFNs. Let $\tilde{\daleth}(s), \tilde{\daleth}(t)$ be the SFs of s and t and $\amalg(s), \amalg(t)$ be the AFs of s and t, respectively. Then,
(1) If $\tilde{\daleth}(s) > \tilde{\daleth}(t)$, then $s > t$
(2) If $\tilde{\daleth}(s) = \tilde{\daleth}(t)$, then

If $\amalg(s) > \amalg(t)$ then $s > t$,
If $\amalg(s) = \amalg(t)$, then $s = t$.

2.3 Some basic AOs related to PFNs

Definition 2.12. Let $\breve{\Upsilon}\varsigma_k = \langle \mu^{\gimel}{}_k, \nu^{\gimel}{}_k, \tau^{\gimel}{}_k \rangle$ be the conglomeration of PFNs. Define $(PFWA) : T^n \to T$ given by

$$\begin{aligned}(PFWA)(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots, \breve{\Upsilon}\varsigma_n) &= \sum_{k=1}^{n} \mathfrak{G}^{\gimel}{}_k \breve{\Upsilon}\varsigma_k \\ &= \mathfrak{G}^{\gimel}{}_1 \breve{\Upsilon}\varsigma_1 \oplus \mathfrak{G}^{\gimel}{}_2 \breve{\Upsilon}\varsigma_2 \oplus \ldots, \mathfrak{G}^{\gimel}{}_n \breve{\Upsilon}\varsigma_n\end{aligned}$$

where T^n is the set of all PFNs, and $\mathfrak{G}^{\gimel} = (\mathfrak{G}^{\gimel}{}_1, \mathfrak{G}^{\gimel}{}_2, \ldots, \mathfrak{G}^{\gimel}{}_n)^T$ is the weight vector (WV) of $(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots \breve{\Upsilon}\varsigma_n)$, s.t. $0 \leqslant \mathfrak{G}^{\gimel}{}_k \leqslant 1$ and $\sum_{k=1}^{n} \mathfrak{G}^{\gimel}{}_k = 1$. Then, the PFWA is the picture fuzzy weighted averaging (PFWA) operator.

We can evaluate PFWA using the operating laws of PFNs, as shown by the preceding theorem.

Theorem 2.13. *Let $\breve{\Upsilon}\varsigma_k = \langle \mu^{\gimel}{}_k, \nu^{\gimel}{}_k \rangle (k = 1, 2, \ldots, n)$ be the conglomeration of PFNs ,we also evaluate the PFWA by*

$$(PFWA)(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots, \breve{\Upsilon}\varsigma_n) = \left\langle (1 - \prod_{k=1}^{n} (1 - \mu^{\gimel}{}_k)^{\mathfrak{G}^{\gimel}{}_k}), \prod_{k=1}^{n} \nu^{\gimel}{}_k^{\mathfrak{G}^{\gimel}{}_k}, \prod_{k=1}^{n} \tau^{\gimel}{}_k^{\mathfrak{G}^{\gimel}{}_k} \right\rangle \tag{1}$$

Example 2.14. Let $\breve{\Upsilon}\varsigma_1 = (0.30, 0.50, 0.15), \breve{\Upsilon}\varsigma_2 = (0.30, 0.50, 0.18)$, and $\breve{\Upsilon}\varsigma_3 = (0.20, 0.40, 0.15)$ be the three PFNs, $w = (0.30, 0.30, 0.40)$ be the WV of $(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \breve{\Upsilon}\varsigma_3)$. We use PFWA operator to aggregate the three PFNs by using (1).

$$\begin{aligned}(\text{PFWA})(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \breve{\Upsilon}\varsigma_3) &= \left\langle (1 - \prod_{k=1}^{3} (1 - \mu^{\gimel}{}_k)^{\mathfrak{G}^{\gimel}{}_k}), \prod_{k=1}^{3} \nu^{\gimel}{}_k^{\mathfrak{G}^{\gimel}{}_k} \right\rangle \\ &= (0.324, 124, 0.123)\end{aligned}$$

Definition 2.15. Let $\breve{\Upsilon}\varsigma_k = \langle \mu^{\gimel}{}_k, \nu^{\gimel}{}_k, \tau^{\gimel}{}_k \rangle$ be the conglomeration of PFN, and (PFWG) : $T^n \to T$, if

$$\begin{aligned}(\text{PFWG})(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots \breve{\Upsilon}\varsigma_n) &= \sum_{k=1}^{n} \breve{\Upsilon}\varsigma_k^{\mathfrak{G}^{\gimel}{}_k} \\ &= \breve{\Upsilon}\varsigma_1^{\mathfrak{G}^{\gimel}{}_1} \otimes \breve{\Upsilon}\varsigma_2^{\mathfrak{G}^{\gimel}{}_2} \otimes \ldots, \breve{\Upsilon}\varsigma_n^{\mathfrak{G}^{\gimel}{}_n}\end{aligned}$$

where $\mathfrak{G}^{\gimel} = (\mathfrak{G}^{\gimel}{}_1, \mathfrak{G}^{\gimel}{}_2, \ldots, \mathfrak{G}^{\gimel}{}_n)^T$ is WV of $(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots, \breve{\Upsilon}\varsigma_n)$, s.t. $0 \leqslant \mathfrak{G}^{\gimel}{}_k \leqslant 1$ and $\sum_{k=1}^{n} \mathfrak{G}^{\gimel}{}_k = 1$. Then, the PFWG is the picture fuzzy weighted geometric (PFWG) operator.

We can evaluate PFWG using the operating laws of PFNs, as shown by the preceding theorem.

Theorem 2.16. *Let $\breve{\Upsilon}\varsigma_k = \langle \mu^{\gimel}{}_k, \nu^{\gimel}{}_k \rangle$ be the conglomeration of PFNs ,we can find PFWG by*

$$(PFWG)(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots \breve{\Upsilon}\varsigma_n) = \left\langle \prod_{k=1}^{n} \mu^{\gimel}{}_k^{\mathfrak{G}^{\gimel}{}_k}, (1 - \prod_{k=1}^{n} (1 - \nu^{\gimel}{}_k)^{\mathfrak{G}^{\gimel}{}_k}), (1 - \prod_{k=1}^{n} (1 - \tau^{\gimel}{}_k)^{\mathfrak{G}^{\gimel}{}_k}) \right\rangle \tag{2}$$

Example 2.17. Let $\breve{\Upsilon}\varsigma_1 = (0.20, 0.30, 0.15), \breve{\Upsilon}\varsigma_2 = (0.30, 0.20, 0.15)$, and $\breve{\Upsilon}\varsigma_3 = (0.40, 0.30, 0.20)$ be the three PFNs, $\mathfrak{G}^{\gimel} = (0.30, 0.30, 0.40)$ be the WV of $(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \breve{\Upsilon}\varsigma_3)$. We use PFWG operator to aggregate the three PFNs by using (2).

$$\begin{aligned}(\text{PFWG})(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \breve{\Upsilon}\varsigma_3) &= \left\langle \prod_{k=1}^{3} \mu^{\gimel}{}_k^{\mathfrak{G}^{\gimel}{}_k}, (1 - \prod_{k=1}^{3} (1 - \nu^{\gimel q}_k)^{\mathfrak{G}^{\gimel}{}_k}), (1 - \prod_{k=1}^{3} (1 - \tau^{\gimel q}_k)^{\mathfrak{G}^{\gimel}{}_k}) \right\rangle \\ &= (0.213, 0.314, 0.342)\end{aligned}$$

2.4 Some deficiencies of PFWA and PFWG operators.

As we all know, PFWA and PFWG operators are utilised to accumulate knowledge in different MCDM issues. Therefore, whenever some values go toward the upper justifications or highest weights, their summed values may imply some absurd results. In this section, we will look at two scenarios.

Case 1: Take two PFNs s.t. $\breve{\Upsilon}\varsigma_1 = (0.001, 0, 0), \breve{\Upsilon}\varsigma_2 = (1, 0, 0)$ with weights $\mathfrak{G}^{\gimel}{}_1 = 0.9$ and $\mathfrak{G}^{\gimel}{}_2 = 0.1$. By (1) and (2) we get

$$\text{PFWA}(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2) = (1, 0, 0)$$

and

$$\text{PFWG}(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2) = (0.002, 0, 0)$$

Case 2: Take two PFNs s.t. $\breve{\Upsilon}\varsigma_1 = (0.001, 0, 0)$, $\breve{\Upsilon}\varsigma_2 = (1, 0, 0)$ with weights $\mathfrak{G}^{\gimel}_1 = 0.1$ and $\mathfrak{G}^{\gimel}_2 = 0.9$. By (1) and (2) we get

$$\text{PFWA}(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2) = (1, 0, 0)$$

and

$$\text{PFWG}(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2) = (0.501, 0, 0)$$

We can see from these data that the PFWA and PFWG operators managed to provide reasonable outcomes in these two circumstances. As a result, in order to address these inadequacies or limitations, we must strengthen the AOs.

3 Some hybrid aggregation operators of PFNs

In this part, we suggest a novel hybrid AOs to fill the gaps left by the PFWA and PFGA operators.

3.1 PFHWAGA operator

Assume that $\breve{\Upsilon}\varsigma_k = \langle \mu^{\gimel}_k, \nu^{\gimel}_k, \tau^{\gimel}_k \rangle (k = 1, 2, \ldots, n)$ is a conglomeration of PFN, and $(PFHWAGA) : T^n \to T$, if

$$(\text{PFHWAGA})(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots, \breve{\Upsilon}\varsigma_n) = \Big(\sum_{k=1}^{n} \mathfrak{G}^{\gimel}_k \breve{\Upsilon}\varsigma_k\Big)^{\sqsupset}\Big(\sum_{k=1}^{n} \breve{\Upsilon}\varsigma_k^{\mathfrak{G}^{\gimel}_k}\Big)^{1-\sqsupset}$$

where, $\sqsupset$ is any real number in $[0, 1]$ and $\mathfrak{G}^{\gimel} = (\mathfrak{G}^{\gimel}_1, \mathfrak{G}^{\gimel}_2, \ldots, \mathfrak{G}^{\gimel}_n)^T$ is WV of $(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots, \breve{\Upsilon}\varsigma_n)$, s.t. $0 \leqslant \mathfrak{G}^{\gimel}_k \leqslant 1$ and $\sum_{k=1}^{n} \mathfrak{G}^{\gimel}_k = 1$. Then, the PFHWAGA is called the PFHWAGA operator. The preceding theorem can be used to find PFHWAGA based on the operating principles of PFNs.

Theorem 3.1. *Let $\breve{\Upsilon}\varsigma_k = \langle \mu^{\gimel}_k, \nu^{\gimel}_k \rangle$ be the conglomeration of PFN ,we can find PFHWAGA by*

(PFHWAGA)$(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots, \breve{\Upsilon}\varsigma_n)$

$$\begin{aligned}
&= \Big(\sum_{k=1}^{n} \mathfrak{G}^{\gimel}_k \breve{\Upsilon}\varsigma_k\Big)^{\sqsupset}\Big(\sum_{k=1}^{n} \breve{\Upsilon}\varsigma_k^{\mathfrak{G}^{\gimel}_k}\Big)^{1-\sqsupset} \\
&= \Big\langle (1 - \prod_{k=1}^{n}(1 - (\mu^{\gimel}_k))^{w_k})^{\sqsupset}(\prod_{k=1}^{n} \mu^{\gimel}{}_k^{w_k})^{1-\sqsupset}, 1 - (1 - (\prod_{k=1}^{n}(\nu^{\gimel}_k)^{w_k}))^{\sqsupset}(\prod_{k=1}^{n}(1 - (\nu^{\gimel}_k))^{w_k})^{1-\sqsupset}, \\
&1 - (1 - (\prod_{k=1}^{n}(\tau^{\gimel}_k)^{w_k}))^{\sqsupset}(\prod_{k=1}^{n}(1 - (\tau^{\gimel}_k))^{w_k})^{1-\sqsupset}\Big\rangle \qquad (3)
\end{aligned}$$

where $\sqsupset$ is any number from $[0, 1]$. $\mathfrak{G}^{\gimel} = (\mathfrak{G}^{\gimel}_1, \mathfrak{G}^{\gimel}_2, \ldots, \mathfrak{G}^{\gimel}_n)^T$ is the WV of $(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots, \breve{\Upsilon}\varsigma_n)$, s.t. $0 \leqslant \mathfrak{G}^{\gimel}_k \leqslant 1$ and $\sum_{k=1}^{n} \mathfrak{G}^{\gimel}_k =$ 1.

Proof. Based on PFWA and PFGA operators and the operational laws of PFSs.

$$\begin{aligned}
&(\text{PFHWAGA})(\breve{\Upsilon}\varsigma_1, \breve{\Upsilon}\varsigma_2, \ldots \breve{\Upsilon}\varsigma_n) \\
&= \Big(\textstyle\sum_{k=1}^{n} \mathfrak{G}^{\gimel}_k \breve{\Upsilon}\varsigma_k\Big)^{\sqsupset}\Big(\textstyle\sum_{k=1}^{n} \breve{\Upsilon}\varsigma_k^{\mathfrak{G}^{\gimel}_k}\Big)^{1-\sqsupset} \\
&= \Big\langle (1 - \textstyle\prod_{k=1}^{n}(1 - \mu^{\gimel}_k)^{\mathfrak{G}^{\gimel}_k}), \textstyle\prod_{k=1}^{n} \nu^{\gimel}{}_k^{\mathfrak{G}^{\gimel}_k}), \textstyle\prod_{k=1}^{n} \tau^{\gimel}{}_k^{\mathfrak{G}^{\gimel}_k})\Big\rangle^{\sqsupset} \\
&\Big\langle (\textstyle\prod_{k=1}^{n} \mu^{\gimel}{}_k^{\mathfrak{G}^{\gimel}_k}, (1 - \textstyle\prod_{k=1}^{n}(1 - \nu^{\gimel}_k)^{\mathfrak{G}^{\gimel}_k})), (1 - \textstyle\prod_{k=1}^{n}(1 - \tau^{\gimel}_k)^{\mathfrak{G}^{\gimel}_k}))\Big\rangle^{1-\sqsupset} \\
&= \Bigg\langle (1 - \textstyle\prod_{k=1}^{n}(1 - (\mu^{\gimel}_k))^{\mathfrak{G}^{\gimel}_k})^{\sqsupset}(\textstyle\prod_{k=1}^{n} \mu^{\gimel}{}_k^{\mathfrak{G}^{\gimel}_k}), 1 - (1 - (\textstyle\prod_{k=1}^{n}(\nu^{\gimel}_k)^{\mathfrak{G}^{\gimel}_k}))^{\sqsupset}(\textstyle\prod_{k=1}^{n}(1 - (\nu^{\gimel}_k))^{\mathfrak{G}^{\gimel}_k})^{1-\sqsupset}\Bigg\rangle, \\
&1 - (1 - (\textstyle\prod_{k=1}^{n}(\tau^{\gimel}_k)^{\mathfrak{G}^{\gimel}_k}))^{\sqsupset}(\textstyle\prod_{k=1}^{n}(1 - (\tau^{\gimel}_k))^{\mathfrak{G}^{\gimel}_k})^{1-\sqsupset}\Bigg\rangle
\end{aligned}$$

Therefore, this complete the proof of (3).

□

Remark. It is feasible to explore the many families of the PFHWAGA operator independently for different values of $\sqsupset \in [0, 1]$. When we consider a particular situation, such as $\sqsupset = 1$, the PFHWAGA operator is converted to the PFWA operator. The PFHWAGA operator is simplified to the PFWG operator if $\sqsupset = 0$. The PFHWAGA operator is the mean of the PFWA and PFWG operators if $\sqsupset = 0.5$.

Example 3.2. Let $\breve{\Upsilon\varsigma}_1 = (0.11, 0.52, 0.10), \breve{\Upsilon\varsigma}_2 = (0.34, 0.16, 0.21)$, and $\breve{\Upsilon\varsigma}_3 = (0.14, 0.38, 0.22)$ be the three PFNs, $\mathfrak{G}^{\daleth} = (0.5, 0.3, 0.2)$ be the WV of $(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \breve{\Upsilon\varsigma}_3)$ and $\sqsupset = 0.5$. We use PFHWAGA operator to aggregate the three PFNs by using (3).

$$
\begin{aligned}
\text{(PFHWAGA)}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \breve{\Upsilon\varsigma}_3) = \Bigg\langle & (1 - \prod_{k=1}^{3}(1-(\mu^{\daleth}{}_k))^{w_k})^{\underline{0.5}}(\prod_{k=1}^{3} \mu^{\daleth}{}_k^{w_k})^{1-0.5}, \\
& 1-(1-(\prod_{k=1}^{3}(\nu^{\daleth}{}_k)^{w_k}))^{0.5}(\prod_{k=1}^{3}(1-(\nu^{\daleth}{}_k))^{w_k})^{1-0.5}, \\
& 1-(1-(\prod_{k=1}^{3}(\tau^{\daleth}{}_k)^{w_k}))^{0.5}(\prod_{k=1}^{3}(1-(\tau^{\daleth}{}_k))^{w_k})^{1-0.5}\Bigg\rangle \\
= & (0.182, 0.267, 0.321)
\end{aligned}
$$

It is clear from the characteristics of the PFWA and PFWG operators that the PFHWAGA operator has idempotency, boundedness, and monotonicity as well.

Theorem 3.3. Let $\breve{\Upsilon\varsigma}_k = \langle \mu^{\daleth}{}_k, \nu^{\daleth}{}_k, \tau^{\daleth}{}_k\rangle (k = 1, 2, \ldots, n)$ is a conglomeration of PFNs. Then,
1. (Idempotency) if $\breve{\Upsilon\varsigma}_k = \breve{\Upsilon\varsigma} = \langle \mu^{\daleth}, \nu^{\daleth}, \tau^{\daleth}\rangle$ for all k, then

$$\text{PFHWAGA}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots, \breve{\Upsilon\varsigma}_n) = \breve{\Upsilon\varsigma}$$

2.(Boundedness) if $\breve{\Upsilon\varsigma}^- = (min(\mu^{\daleth}{}_k), max(\nu^{\daleth}{}_k), max(\tau^{\daleth}{}_k))$ and $\breve{\Upsilon\varsigma}^+ = (max(\mu^{\daleth}{}_k), min(\nu^{\daleth}{}_k), min(\tau^{\daleth}{}_k))$, then we have

$$\breve{\Upsilon\varsigma}^- \leq \text{PFHWAGA}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots, \breve{\Upsilon\varsigma}_n) \leq \breve{\Upsilon\varsigma}^+$$

3.(Monotonicity)If $\breve{\Upsilon\varsigma}_k = \langle \mu^{\daleth}{}_k, \nu^{\daleth}{}_k, \tau^{\daleth}{}_k\rangle$ and $\breve{\Upsilon\varsigma}^*_k = \langle \mu^{\daleth *}_k, \nu^{\daleth *}_k, \tau^{\daleth *}_k\rangle$ are two sets of PFNs. If $\mu^{\daleth}{}_k \geq \mu^{\daleth *}_k, \quad \nu^{\daleth}{}_k \leq \nu^{\daleth *}_k, \quad \tau^{\daleth}{}_k \leq \tau^{\daleth *}_k$ for all k then

$$\text{PFHWAGA}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots, \breve{\Upsilon\varsigma}_n) \geq \text{PFHWAGA}(\breve{\Upsilon\varsigma}^*_1, \breve{\Upsilon\varsigma}^*_2, \ldots, \breve{\Upsilon\varsigma}^*_n)$$

3.2 PFHOWAGA operator

Assume that $\breve{\Upsilon\varsigma}_k = \langle \mu^{\daleth}{}_k, \nu^{\daleth}{}_k, \tau^{\daleth}{}_k\rangle$ is a conglomeration of PFNs, and (PFHOWAGA) : $T^n \to T$, if

$$\text{(PFHOWAGA)}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots \breve{\Upsilon\varsigma}_n) = \Big(\sum_{k=1}^{n} \mathfrak{G}^{\daleth}{}_k \breve{\Upsilon\varsigma}_{\sqsupset(k)}\Big)^{\sqsupset}\Big(\sum_{k=1}^{n} \breve{\Upsilon\varsigma}_{\sqsupset(k)}^{\mathfrak{G}^{\daleth}{}_k}\Big)^{1-\sqsupset}$$

where T^n is the set of all PFNs,$(\sqsupset(1), \sqsupset(2), \ldots, \sqsupset(k))$ is a permutation of $(1, 2, \ldots, n)$ s.t. $\breve{\Upsilon\varsigma}_{\sqsupset(j-1)} \geq \breve{\Upsilon\varsigma}_{\sqsupset(j)}$ for any k, $\sqsupset$ is any real number in the interval $[0, 1]$ and $\mathfrak{G}^{\daleth} = (\mathfrak{G}^{\daleth}{}_1, \mathfrak{G}^{\daleth}{}_2, \ldots, \mathfrak{G}^{\daleth}{}_n)^T$ is WV of $(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots \breve{\Upsilon\varsigma}_n)$, s.t. $0 \leqslant \mathfrak{G}^{\daleth}{}_k \leqslant 1$ and $\sum_{k=1}^{n} \mathfrak{G}^{\daleth}{}_k = 1$. Then, the PFHOWAGA is called the PFHOWAGA operator.
we can also find PFHOWAGA operator by the following theorem.

Theorem 3.4. *Let $\breve{\Upsilon\varsigma}_k = \langle \mu^{\daleth}{}_k, \nu^{\daleth}{}_k\rangle$ is a conglomeration of PFNs, we can find PFHOWAGA by*

$$
\begin{aligned}
\textit{(PFHOWAGA)}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots, \breve{\Upsilon\varsigma}_n) &= \Big(\sum_{k=1}^{n} \mathfrak{G}^{\daleth}{}_k \breve{\Upsilon\varsigma}_{\sqsupset(k)}\Big)^{\sqsupset}\Big(\sum_{k=1}^{n} \breve{\Upsilon\varsigma}_{\sqsupset(k)}^{\mathfrak{G}^{\daleth}{}_k}\Big)^{1-\sqsupset} \\
&= \Bigg\langle (1-\prod_{k=1}^{n}(1-(\mu^{\daleth}{}_{\sqsupset(k)}))^{\mathfrak{G}^{\daleth}{}_k})^{\sqsupset}(\prod_{k=1}^{n} \mu^{\daleth}{}_{\sqsupset(k)}^{\mathfrak{G}^{\daleth}{}_k})^{1-\sqsupset}, \\
& 1-(1-(\prod_{k=1}^{n}(\nu^{\daleth}{}_{\sqsupset(k)})^{\mathfrak{G}^{\daleth}{}_k}))^{\sqsupset}(\prod_{k=1}^{n}(1-(\nu^{\daleth}{}_{\sqsupset(k)}))^{\mathfrak{G}^{\daleth}{}_k})^{1-\sqsupset}, \\
& 1-(1-(\prod_{k=1}^{n}(\tau^{\daleth}{}_{\sqsupset(k)})^{\mathfrak{G}^{\daleth}{}_k}))^{\sqsupset}(\prod_{k=1}^{n}(1-(\tau^{\daleth}{}_{\sqsupset(k)}))^{\mathfrak{G}^{\daleth}{}_k})^{1-\sqsupset}\Bigg\rangle \qquad (4)
\end{aligned}
$$

where $\sqsupset$ is any real number in the interval $[0, 1]$.

Proof. The proof can be made by similar way to proof of Theorem 3.1, so we omit the proof. □

Example 3.5. Let $\ddot{a}_1 = (0.71, 0.12, 0.20), \ddot{a}_2 = (0.54, 0.16, 0.13)$, and $\ddot{a}_3 = (0.57, 0.18, 0.23)$ be the three PFNs, $\mathfrak{G}^{\beth} = (0.5, 0.3, 0.2)$ be the WV of $(\ddot{a}_1, \ddot{a}_2, \ddot{a}_3)$, $\beth = 0.5$.
By SF we rank these PFNs

$$\tilde{\daleth}(\ddot{a}_1) = 0.390$$
$$\tilde{\daleth}(\ddot{a}_2) = 0.250$$
$$\tilde{\daleth}(\ddot{a}_3) = 0.160$$

now $\breve{\Upsilon\varsigma}_1 = \ddot{a}_1, \quad \breve{\Upsilon\varsigma}_2 = \ddot{a}_2, \quad \breve{\Upsilon\varsigma}_3 = \ddot{a}_3$. We use PFHOWAGA operator to aggregate by using (4).

$$\begin{aligned}
\text{(PFHOWAGA)}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \breve{\Upsilon\varsigma}_3) = \Big\langle &(1 - \prod_{k=1}^{3}(1 - (\mu^{\beth}{}_{\beth(k)}))^{w_k})^{0.5}(\prod_{k=1}^{3}\mu^{\beth}{}^{w_k}_{\beth(k)})^{1-0.5},\\
&1 - (1 - (\prod_{k=1}^{3}(\nu^{\beth}{}_{\beth(k)})^{w_k}))^{0.5}(\prod_{k=1}^{3}(1 - (\nu^{\beth}{}_{\beth(k)}))^{w_k})^{1-0.5}\\
&1 - (1 - (\prod_{k=1}^{3}(\tau^{\beth}{}_{\beth(k)})^{w_k}))^{0.5}(\prod_{k=1}^{3}(1 - (\tau^{\beth}{}_{\beth(k)}))^{w_k})^{1-0.5}\Big\rangle\\
&= (0.382, 0.167, 0.102)
\end{aligned}$$

Theorem 3.6. Let $\breve{\Upsilon\varsigma}_k = \langle \mu^{\beth}{}_k, \nu^{\beth}{}_k, \tau^{\beth}{}_k\rangle (k = 1, 2, \ldots, n)$ is a conglomeration of PFNs. Then,
1. (Idempotency) if $\breve{\Upsilon\varsigma}_k = \breve{\Upsilon\varsigma} = \langle \mu^{\beth}, \nu^{\beth}, \tau^{\beth}\rangle$ for all k, then

$$\text{PFHOWAGA}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots, \breve{\Upsilon\varsigma}_n) = \breve{\Upsilon\varsigma}$$

2.(Boundedness) if $\breve{\Upsilon\varsigma}^- = (min(\mu^{\beth}{}_k), max(\nu^{\beth}{}_k), max(\tau^{\beth}{}_k))$ and $\breve{\Upsilon\varsigma}^+ = (max(\mu^{\beth}{}_k), min(\nu^{\beth}{}_k), min(\tau^{\beth}{}_k))$, then we have

$$\breve{\Upsilon\varsigma}^- \leq \text{PFHOWAGA}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots, \breve{\Upsilon\varsigma}_n) \leq \breve{\Upsilon\varsigma}^+$$

3.(Monotonicity)If $\breve{\Upsilon\varsigma}_k = \langle \mu^{\beth}{}_k, \nu^{\beth}{}_k, \tau^{\beth}{}_k\rangle$ and $\breve{\Upsilon\varsigma}^*_k = \langle \mu^{\beth*}_k, \nu^{\beth*}_k, \tau^{\beth*}_k\rangle$ are two sets of PFNs. If $\mu^{\beth}{}_k \geq \mu^{\beth*}_k, \quad \nu^{\beth}{}_k \leq \nu^{\beth*}_k, \quad \tau^{\beth}{}_k \leq \tau^{\beth*}_k$ for all k then

$$\text{PFHOWAGA}(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots, \breve{\Upsilon\varsigma}_n) \geq \text{PFHOWAGA}(\breve{\Upsilon\varsigma}^*_1, \breve{\Upsilon\varsigma}^*_2, \ldots, \breve{\Upsilon\varsigma}^*_n)$$

3.3 Numerical example

To demonstrate the correctness of the aggregated values of the PFHWAGA and PFHOWAGA operations, we consider the first scenario in Section 2.4. If $\beth = 0.5$, we will utilize the PFHWAGA and PFHOWAGA operations.

For case 1, by (3), there is PFHWAGA$(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2) = (0.045, 0, 0)$ which is between PFWA$(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2) = (1, 0, 0)$ and $PFWG(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2) = (0.002, 0, 0)$.

The moderate values are indicated in the above case by new advanced operators. These operators are clearly capable of overcoming the shortcomings of PFWA and PFWG operators. As a result, the PFHWAGA and PFHOWAGA operators are more efficient and acceptable in aggregating data.

4 Multi-criteria decision-making method

Suppose that $\breve{\Upsilon\varsigma} = \{(\breve{\Upsilon\varsigma}_1, \breve{\Upsilon\varsigma}_2, \ldots, \breve{\Upsilon\varsigma}_p)\}$ and $\mathcal{V}^{\mho} = \{\mathcal{V}^{\mho}{}_1, \mathcal{V}^{\mho}{}_2, \ldots, \mathcal{V}^{\mho}{}_q\}$ is the assemblage of alternatives and attributes. Consider $\mathfrak{G}^{\beth}$ be the WV of all criterion, s.t. $\mathfrak{G}^{\beth}{}_j \in [0, 1]$ and $\sum_{j=1}^{n} \mathfrak{G}^{\beth}{}_j = 1$ and $\mathfrak{G}^{\beth}{}_j$ represent the weight of $\mathcal{V}^{\mho}{}_j$. The DM assesses alternatives based on parameters, and the evaluation parameters are in PFNs. Consider $(\breve{\coprod}_{ij})_{p\times q} = \langle \mu^{\beth}{}_{ij}, \nu^{\beth}{}_{ij}\rangle$ is the decision matrix given by the DM.$(\coprod_{ij})$ represent a PFNs for alternative $\breve{\Upsilon\varsigma}_i$ associated with the criterions $\mathcal{V}^{\mho}{}_j$. With this Algorithm 1 some constraints are included s.t.,
1. $\mu^{\beth}{}_{ij}$ and $\nu^{\beth}{}_{ij} \in [0, 1]$
2. $0 \leq \mu^{\beth}{}_A(\mathfrak{G}^{\beth}) + \nu^{\beth}{}_A(\mathfrak{G}^{\beth}) \leq 1, (q \geq 1)$.
We now design Algorithm 1 to tackle the specified issue.

Algorithm 1

Phase i. Obtain the decision matrix from DMs.

$$(\breve{\coprod}_{ij})_{p\times q} = \langle \mu^{\beth}{}_{ij}, \nu^{\beth}{}_{ij}\rangle$$

Phase ii. The decision matrix should be normalised. When we have various kinds of criteria or attributes, such as cost and benefit, we normalise the decision matrix by taking the complement of the cost criteria.
Phase iii. Find β_i^γ = PFHWAGA$(\beta_{i1}^\gamma, \beta_{i2}^\gamma, \ldots \beta_{in}^\gamma)$ or $\beta_i^\gamma = PFHOWAGA(\beta_{i1}^\gamma, \beta_{i2}^\gamma, \ldots \beta_{in}^\gamma)$ for each $i = 1, 2, \ldots, q$.
Phase iv. Evaluate the SFs for all β_i^γ for the collective overall PFNs.
Phase v. Rank all the β_i^γ $(i = 1, 2, \ldots, p)$ according to the score values.

5 MCDM problem related to third-party logistic provider selection

Due to the COVID-19 pandemic's consequences, the e-commerce phenomenon is accelerating, having a huge influence on global supply chains. Thus, logistics management tasks have been elevated in importance in practically every organization that transports physical commodities. There are several methods for businesses to acquire a comparative edge via the out scoured of logistics management processes in today's diversified and incredibly quickly world. Exporters, distributors, and businesses with distribution networks have all demonstrated that turning to third-party logistics (3PL) providers benefits them. 3PL is a term that refers to the process through which a company outsources its warehouse and transportation activities. A 3PL organization can provide stock control, cross-docking, the door of the house distribution, and packaging material. The market for third-party logistics services has accelerated its expansion as a result of the e-commerce boom and expanded reverse logistics activities. The e-commerce trend includes faster, more dependable delivery, increased inventory turnover, and goods staged in forwarding sites near clients. There has been a large surge of 3PL firms offering a range of services to assist in maintaining this very sophisticated supply chain. 3PLs are frequently requested for assistance with e-commerce fulfilment, warehousing, and delivery facilities, and 3PLs invest in technology for both client service and internal usage. Due to the current worldwide problem, the COVID-19 pandemic, the function of e-commerce has been enhanced and expedited.

Due to the features of multidimensional decision-making difficulties, 3PL selection may well be considered a complex MCDM challenge, given the presence of statistical, interpersonal, and numerous factors in the natural decision-making phase. Given the critical nature of sustainable third-party logistics providers, there is a dearth of studies on the 3PL selection challenge in emerging economies. The 3PL sector is growing at a breakneck pace due to the rise of the e-commerce sector. Indeed, the requirement for 3PL services is projected to grow as brands and distributors seek to focus exclusively on their core industries. As a result, they frequently outsource logistical services. In a nutshell, analyzing and choosing optimum third-party logistics providers is a critical component of any business's long-term goals.

Consider a corporation that is looking for the best 3PL provider. Following pre-screening, five 3PLs have been identified for further consideration $\widetilde{\breve{\coprod}}_i$. You must choose between the following four characteristics: (1) $\mathcal{V}^{\mho}{}_1$ = financial stability; (2) $\mathcal{V}^{\mho}{}_2$ = reliability and delivery time; (3) $\mathcal{V}^{\mho}{}_3$ = reputation and (4) $\mathcal{V}^{\mho}{}_4$ = green operation. The DM distributes the attribute weight in the following way: $(0.20, 0.30, 0.10, 0.40)^T$.

Now we will solve the MCDM issue using Algorithm 1. The following sections detail the procedure phases:
Phase i. Evaluating the choice matrix provided by the individual based on PF information.

	$\mathcal{V}^{\mho}{}_1$	$\mathcal{V}^{\mho}{}_2$	$\mathcal{V}^{\mho}{}_3$	$\mathcal{V}^{\mho}{}_4$
$\widetilde{\breve{\coprod}}_1$	(0.200, 0.410, 0.321)	(0.210, 0.250, 0.312)	(0.330, 0.160. 0.111)	(0.160, 0.230, 0.321)
$\widetilde{\breve{\coprod}}_2$	(0.110, 0.220, 0.121)	(0.240, 0.160, 0.321)	(0.170, 0.280, 0.342)	(0.210, 0.290, 0.431)
$\widetilde{\breve{\coprod}}_3$	(0.220, 0.420, 0.321)	(0.230, 0.130, 0.431)	(0.124, 0.230, 0.453)	(0.321, 0.210, 0.321)
$\widetilde{\breve{\coprod}}_4$	(0.260, 0.120, 0.322)	(0.120, 0.320, 0.122)	(0.410, 0.120, 0.423)	(0.100, 0.130, 0.431)

Phase ii. The decision matrix is already in normalized form.
Phase iii. Compute β_i^γ = PFHWAGA$(\beta_{i1}^\gamma, \beta_{i2}^\gamma, \ldots, \beta_{in}^\gamma)$ for each i. Thus we find aggregated PFNs by using Equation (3).

$$\beta_1^\gamma = (0.2180, 0.1270, 0.3212)$$
$$\beta_2^\gamma = (0.3421, 0.2111, 0.3134)$$
$$\beta_3^\gamma = (0.1120, 0.2210, 0.4214)$$
$$\beta_4^\gamma = (0.2420, 0.1990, 0.1345)$$

Phase iv. Evaluate the SFs for all β_i^γ for the collective overall PFNs.

$$\tilde{\daleth}(\beta_1^\gamma) = -0.2302$$
$$\tilde{\daleth}(\beta_2^\gamma) = -0.1824$$
$$\tilde{\daleth}(\beta_3^\gamma) = -0.5304$$
$$\tilde{\daleth}(\beta_4^\gamma) = -0.9150$$

Phase v. Rank all the β_i^γ $(i = 1, 2, 3, 4)$ according to the score values.

$$\beta_2^\gamma \succ \beta_1^\gamma \succ \beta_3^\gamma \succ \beta_4^\gamma$$

and thus β_2^γ ia the most desirable alternative.

6 Conclusion

AOs such as the PFWA and PFWG operators, are significant mathematical tools for integrating PF data. We designed two operators, namely picture fuzzy hybrid weighted arithmetic geometric aggregation (PFHWAGA) operator and picture fuzzy hybrid ordered weighted arithmetic geometric aggregation (PFHOWAGA) operator to address some of the shortcomings of the PFWA and PFWG operators in various real-world problems. Several characteristics of the PFHWAGA and PFHOWAGA operators were discovered. The recommended operators outperform the existing PFN-defined operators. With the use of examples, we enlarged on the proposed operators. Underneath the PF environment, designed operators are more robust and efficient than existing operators. Based on the PFHWAGA and PFHOWAGA operators, we devised an MCDM technique for selecting the best 3PL provider.

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